Some Polynomial-Time Solvable Instances of 3-PARTITION

Byungho Min[1,a)] Shuji Kijima[1,b)]

Abstract:
The 3-PARTITION problem, which is NP-complete, asks whether it is possible to divide given $3m$ numbers into m triples whose sums are the same. We refer to such a triple as the "candidate triple". We prove that the 3-PARTITION problem is solved by deterministic algorithm in polynomial time if all its given numbers are included in two or less candidate triples.

Keywords: NP-complete problem, 3-PARTITION problem

1. Introduction

Definition 1.1 (3-PARTITION problem)**.** *cf.[1]*

INSTANCE : A set A of $3m$ *elements, a bound* $B \in Z$*, and a "size"* $s(a) \in Z$ *for each* $a \in A$*, such that* $\sum_{a \in A} s(a) = mB$*.*

QUESTION : Is it feasible to separate A into m disjoint triples, such that every sum of triple is B.

We will say "*A is 3-partitioned*" if it is possible to separate A into m disjoint triples, such that every sum of triple is B. We name such a triple *candidate triples*.

The number of triples of A is $\binom{3m}{3} = O(m^3)$. Each triple can be checked whether it is a candidate triple in polynomial time. Therefore, the computing all candidate triples just needs polynomial time of $|A|$.

Garey and Johnson [1] proved that 3-PARTITION problem is NP-complete. Hulett et al. [2] proved that this problem is NP-complete even when all elements of A have pairwise distinct sizes. It is challenging problem whether this problem can be solved in polynomial time.

In this note we will show that this problem is solved in polynomial time with the restriction on the maximum number of candidate triples that each element of A is included in.

Theorem 1.2. *If every element of A is included in two or less candidate triples, then we can determine whether or not* $f_{3partition}(A, B) = Y$ *in polynomial time of* $|A|$*.*

2. The proof of Theorem 1.2

For simplicity, we define 3-PARTITION function.

Definition 2.1 (3-PARTITION Function)**.** *The 3-PARTITION function is the function* $f_{3partition} : \{A, B\} \rightarrow \{Y, N\}$ *where* $f_{3partition}(A, B)$ *is "Y" if A can be 3-partitioned, and if not* $f_{3partition}(A, B)$ *is "N".*

1 Kyushu University, Motooka, Fukuoka 819–0395, Japan
a) mmin159ho12@naver.com
b) kijima@inf.kyushu-u.ac.jp

To prove the Theorem 1.2, we make several lemmas.

Lemma 2.2. *If there is* $a \in A$ *which is not included in any candidate triple, then* $f_{3partition}(A, B) = N$*.*

Proof. If $f_{3partition}(A, B) = Y$, then all elements of A should be included in at least one candidate triples. This contradict with the assumption that there is $a \in A$ which is not included in any candidate triples. □

Lemma 2.3. *If there is* $a \in A$ *which is included in just one candidate triple* $\{a, a', a''\}$*, then* $f_{3partition}(A, B) = f_{3partition}(A - \{a, a', a''\}, B)$*.*

Proof. If $f_{3partition}(A - \{a, a', a''\}, B) = Y$, then $f_{3partition}(A, B) = Y$ because only we need to do is add $\{a, a', a''\}$ which is a candidate triple of A to $A - \{a, a', a''\}$. If $f_{3partition}(A, B) = Y$, then $f_{3partition}(A - \{a, a', a''\}, B) = Y$ because 3-PARTITION of A should include $\{a, a', a''\}$ which is sole candidate triple for a. Thus, $f_{3partition}(A, B) = f_{3partition}(A - \{a, a', a''\}, B)$. □

Lemma 2.4. *Supposed that A consists of elements with distinct sizes, and each element in A is included in exactly two candidate triples. Then we can determine whether or not* $f_{3partition}(A, B) = Y$ *in polynomial time of* $|A|$*.*

Proof. First, we convert this 3-PARTITION instance (A, B) into the graph $G = (V(G), E(G))$. $V(G)$ is the set of all candidate triples of A. $E(G)$ is the set of all pairs of two candidate triples that have same element of A. Because all elements in A have pairwise distinct sizes, the degree of each vertex is three.

It is easy to find out whether or not G is a bipartite graph in polynomial time of $|A|$.

Then, we show that $f_{3partition}(A, B) = Y$ if the G is bipartite. $|V_1| = |V_2| = m$ because the degrees of the all vertices are same as three. There are $3|V_1| = 3m$ edges between V_1 and V_2. Each edge means just one element of A and no more than two edges mean same element. Every element of A has corresponding edge of G. Then, we can claim that the set R_1 of candidate triples that is corresponding to V_1 is 3-PARTITION of A because all elements of A are included in at least one candidate triple of R_1 and all candidate triples of R_1 are pairwise distinct. Therefore, we can conclude that $f_{3partition}(A, B) = Y$ if the G is bipartite.

Next, we check that $f_{3partition}(A, B) = N$ if the G is not bipartite. G has odd cycle because it is not bipartite. By using this fact, we will prove that G do not have "Exact Vertex Cover (EVC)" that means the subset of V which each edge in G is at just one vertex in. Let $C = v_{c0}v_{c1}v_{c2}\cdots v_{c(2k)}v_{c0} (k \in N)$ is the one of the odd cycle in G. If we choose k or less vertices from C as members of EVC, then there exists at least one edge in C which do not join any vertex of EVC. So, we should select more than k vertices from C. But, if we pick more than k vertices from C as members of EVC, then there exists at least one edge in C which join two vertex of EVC. Hence, we cannot obtain EVC for G, which means that (A, B) cannot be 3-partitioned. We can know $f_{3partition}(A, B) = N$ if the G is not bipartite.

Finally, we can check whether or not the graph G corresponding to A is bipartite graph in the polynomial time of $|A|$ and we get the value of $f_{3partition}(A, B)$ from it. □

Lemma 2.5. *If all elements of A are included in exactly two candidate triples, then there are at most two elements whose sizes are the same.*

Proof. Supposed that $a_{s1}, a_{s2}, \cdots, a_{sk} \in A$ $(k > 2)$ have same size.

If there are $a_i, a_j \in A$, such that $\{a_{s1}, a_i, a_j\} \in C$, then a_i and a_j have at least k candidate triples because $\{a_{s2}, a_i, a_j\}, \cdots,$ $\{a_{sk}, a_i, a_j\}$ are also candidate triples. This is contradictory to the supposition of lemma.

If there are $a_p \in A$, such that $\{a_{s1}, a_{s2}, a_p\} \in C$, then a_p has at least $k(k-1)/2 \geq 3$ candidate triples because $\{a_{s1}, a_{s3}, a_p\}, \cdots,$ $\{a_{s(k-1)}, a_k, a_p\}$ are also candidate triples. This is contradictory to the supposition of lemma.

If $k = 3$ and $\{a_{s1}, a_{s2}, a_{s3}\} \in C$, then each of them have a candidate triple $\{a_i, a_j, a_k\}$ and need one more candidate triple. But, it is impossible for the identical reason that stated in the former paragraph. Thus, such a case do not exist.

Finally, if $k > 3$ and $\{a_{s1}, a_{s2}, a_{s3}\} \in C$, then a_{s1} has at least $k(k-1)(k-2)/6 \geq 3$ candidate triples because $\{a_{s1}, a_{s2}, a_{s4}\}, \cdots,$ $\{a_{s1}, a_{s(k-1)}, a_{sk}\}$ are also candidate triples. This is contradictory to the supposition of lemma.

Consequentially, there are no more than two elements whose sizes are equal to each other if each element in A has just two candidate triples.

□

Lemma 2.6. *If every element of A is included in exactly two candidate triples, then we can determine whether or not $f_{3partition}(A, B) = Y$ in polynomial time of $|A|$.*

Proof. Three or more elements of A cannot have same size by Lemma 2.5.

Suppose that there are two elements $a_i, a_j \in A$ have same size. Let C_1 and C_2 are the candidate triples that $a_i \in C_1, C_2$. We set $C_1 = \{a_i, a_{p1}, a_{q1}\}$ and $C_2 = \{a_i, a_{p2}, a_{q2}\}$. If $s(a_{p1})$ is equal to $s(a_{p2})$, then a_{q1} have four candidate triples: $\{a_i, a_{p1}, a_{q1}\}$, $\{a_i, a_{p2}, a_{q1}\}$, $\{a_j, a_{p1}, a_{q1}\}$, $\{a_j, a_{p2}, a_{q1}\}$. It is contradictory with the assumption that all element are included in just two candidate triples. This is same to all other combinations: $\{a_{p1}, a_{q2}\}$, $\{a_{q1}, a_{p2}\}$ and $\{a_{q1}, a_{q2}\}$. Thus, $s(a_{p1})$ or $s(a_{q1})$ are not equal to $s(a_{p2})$ or $s(a_{q2})$.

For six elements $a_i, a_j, a_{p1}, a_{q1}, a_{p2}, a_{q2}$, each element has two candidate triples from $C_1 = \{a_i, a_{p1}, a_{q1}\}$, $C_2 = \{a_i, a_{p2}, a_{q2}\}$, $C_3 = \{a_j, a_{p1}, a_{q1}\}$ and $C_4 = \{a_j, a_{p2}, a_{q2}\}$. Each triple of C_1, C_2, C_3 and C_4 has three elements from $a_i, a_j, a_{p1}, a_{q1}, a_{p2}$ and a_{q2}. So, the elements of $\{a_i, a_j, a_{p1}, a_{q1}, a_{p2}, a_{q2}\}$ are detached from the other elements of A. $\{a_i, a_j, a_{p1}, a_{q1}, a_{p2}, a_{q2}\}$ can be 3-partitioned by C_1 and C_4 (or C_2 and C_3). Thus, we can know that $f_{3partition}(A, B) = f_{3partition}(A - \{a_i, a_j, a_{p1}, a_{q1}, a_{p2}, a_{q2}\}, B)$ and the each element of $A - \{a_i, a_j, a_{p1}, a_{q1}, a_{p2}, a_{q2}\}$ also has exactly two candidate triples. By this process, we can eliminate the elements that have same sizes from A without effecting the value of $f_{3partition}(A, B)$.

We repeat aforementioned process until A do not have two elements whose size are same. This end with the polynomial time of $|A|$ since the process can be done in the polynomial time and whenever the process ends the size of A diminish by 6.

Finally, the rest elements of A have pairwise distinct sizes. So, $f_{3partition}(A, B)$ can be determined in polynomial time of $|A|$ by Lemma 2.4. □

Now, we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. Set $A_1 := A$. Supposed that there is $a \in A_1$ which is included in just one candidate triple $\{a, a', a''\}$ of A_1. Then we make $A_1 := A_1 - \{a, a', a''\}$ without any influence on the value of $f_{3partition}(A_1, B)$ by Lemma 2.3. Next, we compute all candidate triples of new A_1. If there is $a_i \in A_1$ which is included in just one candidate triple of A_1, then we do same thing. We repeat this process until A_1 become $\emptyset$, one or more elements of A_1 are not included in any candidate triple or all elements of A_1 are included in just two candidate triples. This repetition must be ended in the polynomial time of $|A|$ because the computation for each process is polynomial and the number of repetition is at most $|A|/3$.

If A_1 become $\emptyset$, we can claim $f_{3partition}(A, B) = Y$.

If there is $a \in A_1$ which is not included in any candidate triple, then we can decide $f_{3partition}(A, B) = N$ by the Lemma 2.2.

If each element of A_1 has exactly two candidate triples, then we can determine whether $f_{3partition}(A, B) = Y$ in the polynomial time of $|A_1|$ which is also polynomial to $|A|$ by the Lemma 2.6.

Therefore, we can determine whether or not $f_{3partition}(A, B) = Y$ in polynomial time of $|A|$ if each element of A is included in two or less candidate triples. □

References

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